Higher Category theory.

Now that we have some basic notions let's try to make sense in ∞-cats. of usual notions from category thy.

If one opens a book in ordinary category theory here is an (incomplete) list of important notions we might want ~~for~~ to make sense for ∞-cat.s as well:

1. initial & final objects. : this one is easy:

Def'n: An object $X \in \mathcal{C}$ (∞-cat.) is

- initial if for all $Y \in \mathcal{C}$: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq *$.

recall this is in $\underline{Top}$.

- final if for all $Y \in \mathcal{C}$: $\mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq *$.

2. limits and colimits.

(i) (sleek ~~way~~ way). let $F: K \to \mathcal{C}$ be a diagram. (i.e. functor from K an ∞-cat., more generally even simplicial set). Consider: $\Delta_F: \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ diagonal map, then $\mathrm{Colim}\, F$ is the object in $\mathcal{C}$ that corepresents.

$$\mathcal{C} \to \boxed{?}$$
$$X \mapsto \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(F, \Delta_F(X))$$, i.e.

$$\mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(F, \Delta_F(X)) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{Colim}\, F, X). \qquad (\Delta)$$

but we need (Δ) functorially, so what goes in the place of $\boxed{?}$

Say $\boxed{?} = \mathrm{Spc}$, then $\underline{Q}$: How to write $\mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(F, \Delta_F(-))$ as a functor?

(ii) use 1. + an ∞-category we can construct out of F.

<u>Def'n:</u> Let $X \in \mathcal{C}$ the <u>under category</u> w.r.t. X be the pullback (in simplicial sets)

$$\begin{tikzcd} \mathcal{C}^{X/} \arrow[r] \arrow[d, "\{X\}\times id_{\mathcal{C}}"] \arrow[dr, phantom, "\lrcorner", very near start] & Fun([1], \mathcal{C}) \, (= Map_{Sets_\Delta}(\Delta^1, \mathcal{C})). \arrow[d, "ev_0"] \\ \mathcal{C} \arrow[r] & Fun(\{0,1\}, \mathcal{C}) \end{tikzcd}$$

More generally, for $F: K \to \mathcal{C}$ we define $\mathcal{C}^{F/}$ by the pullbacks:

$$\begin{tikzcd} \mathcal{C}^{F/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fun(K, \mathcal{C})^{\tilde{F}/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fun(K \times [1], \mathcal{C}). \arrow[d] \\ \mathcal{C} \arrow[r, "\Delta_F"] & \{\bar{F}\} \times Fun(K, \mathcal{C}) \arrow[r] & Fun(K \times \{0,1\}, \mathcal{C}) \end{tikzcd}$$

($\bar{F}$ is the image of F in $Fun(K, \mathcal{C})_0$).

<u>Prop:</u> For any diagram $p: K \to \mathcal{C}$ the simplicial set $\mathcal{C}^{F/}$ is an ∞-category.

<u>Pf idea:</u> (just in the case $F: \Delta^0 \to \mathcal{C}$, i.e. $X \in \mathcal{C}$).

$$\begin{tikzcd} Hom_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{X/} \arrow[d, "p"] \\ \{Y\} \arrow[r] & \mathcal{C} \end{tikzcd}$$

. Actually, the map p is a left fibration, i.e. lifts against $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i < n$.

Then $\mathcal{C}^{X/} \to \mathcal{C} \to *$ is an inner fibration. ■

(e.g.
$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & \mathcal{C}^{X/} \arrow[d] \\ \Delta^1 \arrow[r] & \mathcal{C} \end{tikzcd}$$
says.
$$\begin{tikzcd} X \arrow[d] & & X \arrow[d] \arrow[dr] & \\ Y \arrow[r] & Z \arrow[r, squiggly] & Y \arrow[r] & Z \end{tikzcd}$$
.).

<u>Def'n</u>: A <u>colimit</u> of a diagram $F: K \to \mathcal{C}$ is an <u>initial</u> object in $\mathcal{C}^{F/}$.

<u>Rk</u>: (1) ~~the space of initial (or final) objects is contractible, or~~
an initial object is unique up to a contractible space of choices, i.e. $\mathcal{C}_{X/-} \to \mathcal{C}$ is a trivial Kan fibration.
Follows from: $\mathcal{C}_{X/-} \to \mathcal{C}$ is a left fibration, + $\forall\, Y \in \mathcal{C}$ the fiber $\mathcal{C}_{X/-} \times_{\mathcal{C}} \{Y\} \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is contractible. [HTT Lemma 2.1.3.4.]

(2) Our def'n of $\mathcal{C}^{X/-}$ (or of $\mathcal{C}^{-/X}$) is a slightly diff. than the initial definition in [HTT §1.2.9], denoted $\mathcal{C}_{X/-}$, though $\mathcal{C}^{X/-}$ & $\mathcal{C}_{X/-}$ turns out to be equivalent as ∞-categories. [HTT 4.2.1.5.]

(iii) colimits of all diagrams of shape K, i.e. $\mathrm{Fun}(K, \mathcal{C})$ this is defined a the left adjoint below:

$$\begin{tikzcd} \mathrm{Fun}(K, \mathcal{C}) \arrow[r, bend left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\Delta_K(-)"] \end{tikzcd}.$$

This leads us to the discussion of a third important concept in category theory.

3. <u>Adjunction functors</u>: in usual cat. thy we have the data:

$$F: C \rightleftarrows D : G \quad (\perp) \qquad + \text{ natural transformations:}$$

$$\mathrm{id}_C \Rightarrow G \circ F \qquad \& \qquad F \circ G \Rightarrow \mathrm{id}_D.$$

Problem: We don't have a 2-category structure on Cat_∞, i.e. we killed all non-invertible natural transformations in its definition. It would be great to be able to formulate this data using only Cat_∞.

4. In Talk 4 we discussed the construction of an associative composition map $\circ$ on the category Top. However, what about promoting this to a functor of ∞-categories:

$$\mathrm{Hom}_{\mathcal{C}}(-,-): \mathcal{C}^{op} \times \mathcal{C} \to Spc.$$

(not just into $hSpc \simeq$ Top.)

These two problems are subtle in ∞-categories, but the notion of Cartesian & coCartesian fibrations helps answer both problems:

(i) write functors between ∞-categories, specially if the target is Spc or Cat_∞.

(ii) transform 2-categorical data into 1-categorical data.

Fibrations & Grothendieck construction:

We are interested in the following picture:

$$\begin{array}{lll} \mathcal{C} & & X \xrightarrow{\tilde{f}} Y \\ p \downarrow & \text{map of } \infty\text{-cats.} & \\ \mathcal{D} & & d \xrightarrow{f} d' \end{array}$$

where we want:

- $\forall d \in \mathcal{D}$, $\mathcal{C}_d := \underbrace{\mathcal{C} \times_{\mathcal{D}} \{d\}}_{p^{-1}(d) =}$ is an ∞-cat

- $\forall$ morphism $d \xrightarrow{f} d'$ a functor $f^*: \mathcal{C}_{d'} \to \mathcal{C}_d$, i.e. for each $Y \in \mathcal{C}_{d'}$ give the data of a morphism $\tilde{f}: X \to Y$ s.t. $p(\tilde{f}) = f$, i.e. $f^* Y = X$.

Q: How to chose this data $\tilde{f}: X \to Y$?

Heuristically, we want to say that for any object Z in $\mathcal{C}$

(the data of a map $Z \to X$) should be recovered from the data of a map from Z into Y in $\mathcal{C}$ and into d and d' and $p(Z)$ in $\mathcal{D}$.

$d = p(X)$, $d' = p(Y)$.

After a bit of thought one writes that $\tilde{f}$ should be such that the canonical morphism:

$$(\Delta) \quad \mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, Y) \times_{\mathrm{Hom}_{\mathcal{D}}(p(Z), p(Y))} \mathrm{Hom}_{\mathcal{D}}(p(Z), p(X)).$$

is an isomorphism in Spc.

Exercise: (Δ) being an isom. is equivalent to asking that $\tilde{f}$ is a final object in $\mathcal{C}_{/Y} \times_{\mathcal{D}_{/p(Y)}} \mathcal{D}_{/\bar{f}}$.

Def'n: A morphism $f: X \to Y$ in $\mathcal{C}$ satisfying (Δ) is called p-Cartesian, and we say f is a p-Cartesian lift of $p(f)$ in $\mathcal{D}$.

Def'n: A functor $p: \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration if for every $Y \in \mathcal{C}$ and $\bar{f}: \bar{X} \to p(Y) (= \bar{Y})$ there exists a p-Cartesian lift of $\bar{f}$.

Before giving examples of Cartesian (or coCartesian) fibrations we mention the straightening / unstraightening result, which confirms that the notion of fibrations capture the data of functors into the ∞-category of ∞-cats.

Let $\mathrm{Cart}(\mathcal{C})$ be the subcategory of $\mathrm{Cat}_\infty^{\mathcal{C}/-}$ generated by Cartesian fibrations where morphisms take p-Cartesian morphisms

$$\begin{tikzcd} \mathcal{D} \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D}' \arrow[dl, "p'"] \\ & \mathcal{C} & \end{tikzcd}$$

to p'-Cartesian morphisms.

The following is Thm. 3.2.0.1. [HTT].

<u>Thm</u>: For any ∞-category $\mathcal{C}$ one has an equivalence:

$$St : \mathrm{Cart}(\mathcal{C}) \overset{\simeq}{\rightleftarrows} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty) : Un,$$

where informally

- the <u>straightening functor</u> is given by
$$St(\mathcal{D} \xrightarrow{p} \mathcal{C}) : \mathcal{C}^{op} \to \mathrm{Cat}_\infty.$$
$$X \mapsto p^{-1}(X)$$
$$X \xrightarrow{f} Y \mapsto f^* : p^{-1}(Y) \to p^{-1}(X).$$

- <u>unstraightening functor</u> is given by:
$Un(\mathcal{C}^{op} \xrightarrow{F} \mathrm{Cat}_\infty)$ is the ∞-category w/
  - objects: (X, D), $X \in \mathcal{C}$, $D \in F(X)$.
  - morphisms. $(X, D) \to (Y, D')$ is $f : X \to Y$ + $\alpha : F(f)(D') \to D$.

<u>Rk</u>: (i) (St, Un) forms an adjoint pair.

(ii) Un is sometimes called the Grothendieck construction, in analogy to the similar construction in the thy of fibered categories.